Reasoning with strict symmetric monoidal categories in Agda

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Monoidal Categories

Describe structures that can be composed in sequence and in parallel.

Definition

A category **C** is *monoidal* if it is equipped with a functor $\otimes : \mathbf{C} \times \mathbf{C}$, and morphisms:

- left unit $\lambda_A : 1 \otimes A \to A$
- right unit $\rho_A : A \otimes 1 \to A$
- associativity $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$

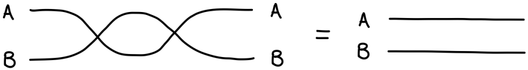
satisfying the triangle and pentagon equalities.

Symmetric Monoidal Categories (SMCs)

Definition

A monoidal category is *symmetric* if it is equipped with a swap operation $\sigma_{A,B} : A \otimes B \to B \otimes A$, such that $\sigma_{A,B} \fatsemi \sigma_{B,A} = 1_{A \otimes B}$ and the hexagon equality holds.

Double swap:



Hexagon equality:

$$\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes B \otimes C & \xrightarrow{\sigma_{A,B \otimes C}} & (B \otimes C) \otimes A \\
\downarrow{\scriptstyle \sigma_{A,B} \otimes 1_C} & & & & \downarrow{\scriptstyle \alpha_{B,C,A}} \\
(B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes A \otimes C & \xrightarrow{1_B \otimes \sigma_{A,C}} & B \otimes C \otimes A
\end{array}$$

SMCs in Agda

A category in which all morphisms are invertible:

```
data Ob : Set where
  one : Ob
  _⊗_ : Ob → Ob → Ob
  var : ℕ → Ob

data _↔_ : Ob → Ob → Set where
  id : (A : Ob) → A ↔ A
  _⨾_ : A ↔ B → B ↔ C → A ↔ C
  _⊗_ : A ↔ B → C ↔ D → (A ⊗ C) ↔ (B ⊗ D)
  sym : A ↔ B → B ↔ A
  swap⊗ : (A B : Ob) → A ⊗ B ↔ B ⊗ A
```

Reasoning with morphisms

The 2-level structure defines equations between morphisms.

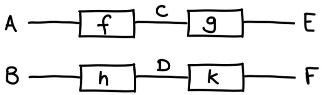
$$
\begin{aligned}
&\text{data } \_\Leftrightarrow\_ : (A \leftrightarrow B) \to (A \leftrightarrow B) \to \mathsf{Set} \text{ where}\\
&\quad \mathsf{id} : \{c : A \leftrightarrow B\} \to c \Leftrightarrow c\\
&\quad \mathsf{sym} : \{c\ d : A \leftrightarrow B\} \to c \Leftrightarrow d \to d \Leftrightarrow c\\
&\quad \_\fatsemi\_ : \{c\ d\ e : A \leftrightarrow B\} \to c \Leftrightarrow d \to d \Leftrightarrow e \to c \Leftrightarrow e
\end{aligned}
$$

(+ constructors for $\fatsemi$ and $\otimes$ of terms)

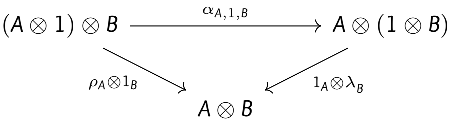
For example, we can specify that $\otimes$ is a functor:

$$\mathsf{id}{\otimes}\mathsf{id} : (\mathsf{id}\ A \otimes \mathsf{id}\ B) \Leftrightarrow \mathsf{id}\ (A \otimes B)$$

$$
\begin{aligned}
&\mathsf{hom}{\otimes} : \{f : A \leftrightarrow C\}\{g : C \leftrightarrow E\}\{h : B \leftrightarrow D\}\{k : D \leftrightarrow F\}\\
&\qquad \to (f \fatsemi g) \otimes (h \fatsemi k) \Leftrightarrow (f \otimes h) \fatsemi (g \otimes k)
\end{aligned}
$$



Triangle equality



triangle : unit⊗r A ⊗ id B ⇔ assoc⊗ {A}{one}{B} ⨾ (id A ⊗ unit⊗l B)

Pentagon equality

$$
\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A\otimes B,C,D}} & (A \otimes B) \otimes C \otimes D & \xrightarrow{\alpha_{A,B,C\otimes D}} & A \otimes B \otimes C \otimes D \\
\downarrow{\scriptstyle \alpha_{A,B,C}\otimes 1_D} & & & & \uparrow{\scriptstyle 1_A \otimes \alpha_{B,C,D}} \\
(A \otimes B \otimes C) \otimes D & & \xrightarrow{\qquad\alpha_{A,B\otimes C,D}\qquad} & & A \otimes (B \otimes C) \otimes D
\end{array}
$$

pentagon : assoc⊗ {A ⊗ B} {C} {D} ⨾ assoc⊗ {A} {B} {C ⊗ D}
⇔ assoc⊗ {A} {B} {C} ⊗ id D ⨾ assoc⊗ {A} {B ⊗ C} {D} ⨾ id A ⊗ assoc⊗ {B} {C} {D}

Coherence isomorphisms on morphisms

To express associativity and unit laws on morphisms, we have to include explicit equations on objects to fix up the types:

$$\text{assoc}\otimes\text{m'} : \{f : A \leftrightarrow B\}\{g : B \leftrightarrow C\}\{h : C \leftrightarrow D\} \to (f \otimes g) \otimes h \Leftrightarrow \text{assoc}\otimes \mathbin{⨟} f \otimes g \otimes h \mathbin{⨟} \text{sym assoc}\otimes$$

$$\text{unit}\otimes\text{ml'} : \{f : A \leftrightarrow B\} \to \text{id one} \otimes f \Leftrightarrow \text{unit}\otimes\text{l}\ A \mathbin{⨟} f \mathbin{⨟} \text{sym}\ (\text{unit}\otimes\text{l}\ B)$$

$$\text{unit}\otimes\text{mr'} : \{f : A \leftrightarrow B\} \to f \otimes \text{id one} \Leftrightarrow \text{unit}\otimes\text{r}\ A \mathbin{⨟} f \mathbin{⨟} \text{sym}\ (\text{unit}\otimes\text{r}\ B)$$

For reasoning with weak MCs, all of the structural equivalences have to be explicit.

Strict SMCs

Definition

In a strict SMC, associativity and unit morphisms are the identity.

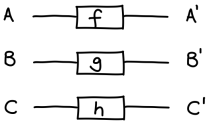
Lemma

In a strict SMC, triangle and pentagon equalities are the identity.

Remark: both paths are the identity, but also the filling of the commutative diagrams.

String Diagrams

In string diagrams the strictness property is trivially satisfied. For example, $(f \otimes g) \otimes h = f \otimes g \otimes h$:



String diagrams depict equivalence classes of morphisms of monoidal categories.

Reasoning with strict SMCs in Agda

- only computational content is in the swap operations (morphisms are permutations)
- ideally we would only talk about swaps in proofs
- implicit structural equalities have to be explicit in Agda

$$\text{assoc}{\otimes}\text{m'} : \{f : A \leftrightarrow B\}\{g : B \leftrightarrow C\}\{h : C \leftrightarrow D\} \to (f \otimes g) \otimes h \Leftrightarrow \text{assoc}{\otimes} \fatsemi f \otimes g \otimes h \fatsemi \text{sym assoc}{\otimes}$$

Agda's Rewrite Rules[1]

Adding user-specified definitional equalities to the theory.

```
data ℕ : Set where
  zero : ℕ
  suc : ℕ → ℕ

_+_ : ℕ → ℕ → ℕ
zero + n = n
(suc m) + n = suc (m + n)

unit+ : (a : ℕ) → a + zero ≡ a
unit+ zero = refl
unit+ (suc a) = cong suc (unit+ a)

assoc+ : (a b c : ℕ) → (a + b) + c ≡ a + b + c
assoc+ zero b c = refl
assoc+ (suc a) b c = cong suc (assoc+ a b c)

{-# REWRITE unit+ #-}

{-# REWRITE assoc+ #-}
```

[1] Cockx, "Type Theory Unchained: Extending Agda with User-Defined Rewrite Rules".

Rewrite Example

```
data Vec (A : Set) : ℕ → Set where
  [] : Vec A zero
  _::_ : ∀{n} → (x : A) (xs : Vec A n) → Vec A (suc n)
```

```
_++_ : {A : Set}{m n : ℕ} →
        Vec A m → Vec A n → Vec A (m + n)
[] ++ vs' = vs'
(v :: vs) ++ vs' = v :: (vs ++ vs')
```

```
assoc++ : {X : Set} → {k l m : ℕ} (as : Vec X k)(bs : Vec X l)(cs : Vec X m) →
          (as ++ bs) ++ cs ≡ as ++ (bs ++ cs)
```

It typechecks! Even though:

- (as ++ bs) ++ cs : Vec X ((k ++ l) ++ m)
- as ++ (bs ++ cs) : Vec X (k ++ (l ++ m))

Use rewrite rules on arbitrary relation

I can use rewrite rules on relations that I have specified myself.

- idea: work with equivalence classes of the relation
- choose one representative and rewrite everything else to it (e.g. associate to the right)

Plan: use it to declare coherence isomorphisms of SMCs as definitional equalities.

- extract computationally relevant part of a proof
- proof in Agda to look like the paper one

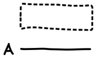
Rewrite equations on objects

$\mathsf{assoc}\otimes : (A \otimes B) \otimes C \leftrightarrow A \otimes B \otimes C$

{-# REWRITE assoc⊗ #-}

$\mathsf{unit}\otimes\mathsf{l} : (A : \mathsf{Ob}) \to (\mathsf{one} \otimes A) \leftrightarrow A$

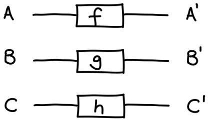
$\mathsf{unit}\otimes\mathsf{r} : (A : \mathsf{Ob}) \to (A \otimes \mathsf{one}) \leftrightarrow A$



{-# REWRITE unit⊗l unit⊗r #-}

Equations on morphisms

$\text{assoc}\otimes\text{mr} : \{f : A \leftrightarrow B\}\{g : B \leftrightarrow C\}\{h : C \leftrightarrow D\} \to (f \otimes g) \otimes h \Leftrightarrow f \otimes g \otimes h$



$\text{unit}\otimes\text{ml} : \{f : A \leftrightarrow B\} \to \text{id one} \otimes f \Leftrightarrow f$
$\text{unit}\otimes\text{mr} : \{f : A \leftrightarrow B\} \to f \otimes \text{id one} \Leftrightarrow f$

Add these equations as definitional equalities, too!

```
{-# REWRITE assoc⊗mr unit⊗ml unit⊗mr #-}
```

Strict SMCs in Agda

Even stronger: *we* can now strictify the category, by declaring...

$$\mathsf{assoc}{\otimes}{=}\mathsf{id} : \mathsf{assoc}{\otimes}\ \{A\}\{B\}\{C\} \Leftrightarrow \mathsf{id}\ (A \otimes B \otimes C)$$
$$\mathsf{unit}{\otimes}\mathsf{l}{=}\mathsf{id} : \mathsf{unit}{\otimes}\mathsf{l}\ A \Leftrightarrow \mathsf{id}\ A$$
$$\mathsf{unit}{\otimes}\mathsf{r}{=}\mathsf{id} : \mathsf{unit}{\otimes}\mathsf{r}\ A \Leftrightarrow \mathsf{id}\ A$$

...and immediately rewriting by these equations.

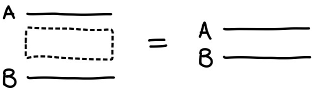
Additionally, *we* rewrite by functoriality of $\otimes$, e.g.

$$\mathsf{id}{\otimes}\mathsf{id} : (\mathsf{id}\ t1\ \otimes\!\downarrow \mathsf{id}\ t2) \Leftrightarrow \mathsf{id}\ (t1 \otimes t2)$$

Triangle Equality in a strict SMC

$$\begin{array}{ccc} (A \otimes 1) \otimes B & \xrightarrow{\alpha_{A,1,B}} & A \otimes (1 \otimes B) \\ & {\scriptstyle \rho_A \otimes 1_B} \searrow \qquad \swarrow {\scriptstyle 1_A \otimes \lambda_B} & \\ & A \otimes B & \end{array}$$

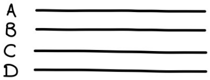
```
triangle : {A B : Ob} → unit⊗r A ⊗ id B ⇔ assoc⊗r {A}{one}{B} ⨾ (id A ⊗ unit⊗l B)
triangle = id
```



Pentagon Equality in a strict SMC

$$\begin{array}{ccccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A \otimes B, C, D}} & (A \otimes B) \otimes C \otimes D & \xrightarrow{\alpha_{A, B, C \otimes D}} & A \otimes B \otimes C \otimes D \\
\downarrow{\scriptstyle \alpha_{A,B,C} \otimes 1_D} & & & & \uparrow{\scriptstyle 1_A \otimes \alpha_{B,C,D}} \\
(A \otimes B \otimes C) \otimes D & & \xrightarrow{\alpha_{A, B \otimes C, D}} & & A \otimes (B \otimes C) \otimes D
\end{array}$$

pentagon : {A B C D : Ob} →
assoc⊗r {A ⊗ B} {C} {D} ⨾ assoc⊗r {A} {B} {C ⊗ D}
⇔ assoc⊗r {A} {B} {C} ⊗ id D ⨾ assoc⊗r {A} {B ⊗ C} {D} ⨾ id A ⊗ assoc⊗r {B} {C} {D}
pentagon = id



Hexagon in a strict SMC

$$
\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes B \otimes C & \xrightarrow{\sigma_{A,B\otimes C}} & (B \otimes C) \otimes A \\
\downarrow{\scriptstyle \sigma_{A,B}\otimes 1_C} & & & & \downarrow{\scriptstyle \alpha_{B,C,A}} \\
(B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes A \otimes C & \xrightarrow{1_B\otimes\sigma_{A,C}} & B \otimes C \otimes A
\end{array}
$$

What to use it for?

I'm interested in rig categories[2]:

- Structural foundation for the semantics of quantum computation.
- Contain two monoidal structures $(\otimes, 1)$ and $(\oplus, 0)$.
- Distributive law between them: $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$.
- a lot of coherence conditions! Including a lot about structural equivalences.

[2] Heunen and Kaarsgaard, "Quantum information effects".

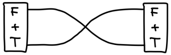
Two versions of not[3]

Type of booleans:

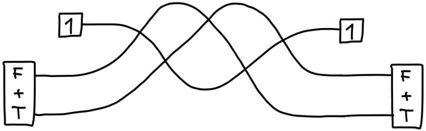
bool = one ⊕ one

Two implementations of the not operation:

not1 : bool ↔ bool
not1 = swap⊕ one one



not2 : bool ↔ bool
not2 = sym (unit⊗l (one ⊕ one))
  ⨾ swap⊗ one (one ⊕ one)
  ⨾ swap⊕ one one ⊗↓ id one
  ⨾ swap⊗ (one ⊕ one) one
  ⨾ unit⊗l (one ⊕ one)



[3] Carette and Sabry, "Computing with Semirings and Weak Rig Groupoids".

Not is not

```
same-not : not2 ⇔ not1
same-not =
  swap⊗-nat {one ⊕ one}{one ⊕ one}{one}{one}
            {swap⊕ one one}{id one}
            ⨾↓ id {c = swap⊗ (one ⊕ one) one}
  ⨾ (id ⨾↓ swap⊗2 {one}{one ⊕ one}
  ⨾ unit⨾↓r {one ⊗ (one ⊕ one)}
  ⨾ unit⊗↓l (swap⊕ one one))
```

```
negEx : NOT₂ ⇔ NOT₁
negEx = uniti⋆l ⊙ (Pi0.swap⋆ ⊙ ((Pi0.swap₊ ⊗ id↔) ⊙ (Pi0.swap⋆ ⊙ unite⋆l)))
    ⇔⟨ id⇔ ⊡ assoc⊙l ⟩
  uniti⋆l ⊙ ((Pi0.swap⋆ ⊙ (Pi0.swap₊ ⊗ id↔)) ⊙ (Pi0.swap⋆ ⊙ unite⋆l))
    ⇔⟨ id⇔ ⊡ (swapl⋆⇔ ⊡ id⇔) ⟩
uniti⋆l ⊙ (((id↔ ⊗ Pi0.swap₊) ⊙ Pi0.swap⋆) ⊙ (Pi0.swap⋆ ⊙ unite⋆l))
  ⇔⟨ id⇔ ⊡ assoc⊙r ⟩
uniti⋆l ⊙ ((id↔ ⊗ Pi0.swap₊) ⊙ (Pi0.swap⋆ ⊙ (Pi0.swap⋆ ⊙ unite⋆l)))
  ⇔⟨ id⇔ ⊡ (id⇔ ⊡ assoc⊙l) ⟩
uniti⋆l ⊙ ((id↔ ⊗ Pi0.swap₊) ⊙ ((Pi0.swap⋆ ⊙ Pi0.swap⋆) ⊙ unite⋆l))
  ⇔⟨ id⇔ ⊡ (id⇔ ⊡ (linv⊙l ⊡ id⇔)) ⟩
uniti⋆l ⊙ ((id↔ ⊗ Pi0.swap₊) ⊙ (id↔ ⊙ unite⋆l))
  ⇔⟨ id⇔ ⊡ (id⇔ ⊡ idl⊙l) ⟩
uniti⋆l ⊙ ((id↔ ⊗ Pi0.swap₊) ⊙ unite⋆l)
  ⇔⟨ assoc⊙l ⟩
(uniti⋆l ⊙ (id↔ ⊗ Pi0.swap₊)) ⊙ unite⋆l
  ⇔⟨ unitil⋆⇔l ⊡ id⇔ ⟩
(Pi0.swap₊ ⊙ uniti⋆l) ⊙ unite⋆l
  ⇔⟨ assoc⊙r ⟩
Pi0.swap₊ ⊙ (uniti⋆l ⊙ unite⋆l)
  ⇔⟨ id⇔ ⊡ linv⊙l ⟩
Pi0.swap₊ ⊙ id↔
  ⇔⟨ idr⊙l ⟩
Pi0.swap₊ □
```

Confluence checking

- Agda has a local-confluence-check pragma for rewrite rules
- this does not interact well with rewrite rules that typecheck because of other rewrite rules:

$$\mathsf{assoc}{\otimes}\mathsf{mr} : \{f : A \leftrightarrow B\}\{g : B \leftrightarrow C\}\{h : C \leftrightarrow D\} \rightarrow (f \otimes g) \otimes h \Leftrightarrow f \otimes g \otimes h$$

- check confluence by hand…

Summary

- strict SMC contain a lot of trivial structural coherence isomorphisms
- with Agda's rewrite rules these can be implicit in the formalisation
- can extract the computational interesting part of a proof

Some future ideas:

- apply to other flavours of MC (e.g. braided)
- explore rewriting of setoid equalities in Agda
- rewriting heterogeneous equalities?

Thank you for your attention!

Carette, Jacques and Amr Sabry. "Computing with Semirings and Weak Rig Groupoids". In: *Programming Languages and Systems - 25th European Symposium on Programming, ESOP 2016, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2016, Eindhoven, The Netherlands, April 2-8, 2016, Proceedings*. Ed. by Peter Thiemann. Vol. 9632. Lecture Notes in Computer Science. Springer, 2016, pp. 123–148. DOI: 10.1007/978-3-662-49498-1_6. URL: https://doi.org/10.1007/978-3-662-49498-1_6.

Cockx, Jesper. "Type Theory Unchained: Extending Agda with User-Defined Rewrite Rules". In: *25th International Conference on Types for Proofs and Programs, TYPES 2019, June 11-14, 2019, Oslo, Norway*. Ed. by Marc Bezem and Assia Mahboubi. Vol. 175. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019, 2:1–2:27. DOI: 10.4230/LIPICS.TYPES.2019.2. URL: https://doi.org/10.4230/LIPIcs.TYPES.2019.2.

Heunen, Chris and Robin Kaarsgaard. "Quantum information effects". In: *Proc. ACM Program. Lang.* 6.POPL (2022), pp. 1–27. DOI: 10.1145/3498663. URL: https://doi.org/10.1145/3498663.