A Category of Surface-Embedded Graphs

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- string diagrams syntaxes for monoidal categories
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here: vertices represent generators, edges represent wires

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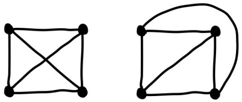
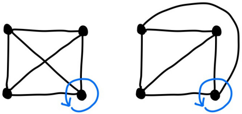
- monoidal theories may require non-trivial topology of graphs in particular non-symmetric theories
- example: string diagrams for quantum processes
- easiest case: plane graph embeddings



- working at the level of the *embedding*

Representing Graph Embeddings

Rotation Systems fix order of edges around vertices



Theorem

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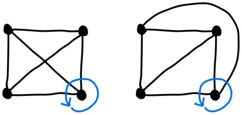

Theorem

Rotation systems uniquely determine a graph embedding.

plan: construct a category of graphs, then add rotation information

Diagram Rewriting

$$
\begin{tikzcd}
L \arrow[d] & B \arrow[l] \arrow[r] \arrow[d] & R \arrow[d] \\
G \arrow[ur, phantom, "\urcorner", very near start] & G \setminus L \arrow[l] \arrow[r] & G[R/L] \arrow[ul, phantom, "\ulcorner", very near start]
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- rewrite rules are $L \Rightarrow R$, with common boundary B
- double-pushout diagram, all maps are embeddings
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- $C = G \setminus L$: context with a hole
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- need: pushouts, pushout complements, notion of embedding

Category of Graphs

We start from the usual category of graphs:

- graphs are $E \underset{t}{\overset{s}{\rightrightarrows}} V$
- morphisms are pairs of edge map f_E and vertex map f_V s.t.

$$\begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "s"'] & E' \arrow[d, "s'"] \\ V \arrow[r, "f_V"'] & V' \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "t"'] & E' \arrow[d, "t'"] \\ V \arrow[r, "f_V"'] & V' \end{tikzcd}$$

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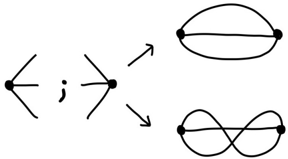
Disclaimer

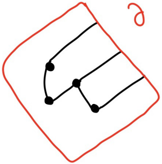
(Almost) all graphs are drawn undirected in this presentation.

Open Graphs

- have to encode inputs and outputs of the diagrams
- different approaches: open graphs, representative vertices, cospans
- morphisms for open graphs don't preserve the surface:



Boundary Vertices



- identify the "outside" of a graph
- attach input and output edges to this region

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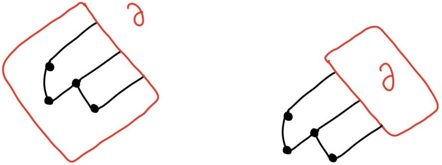

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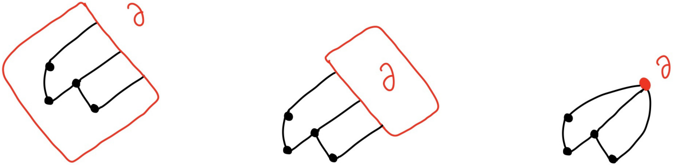

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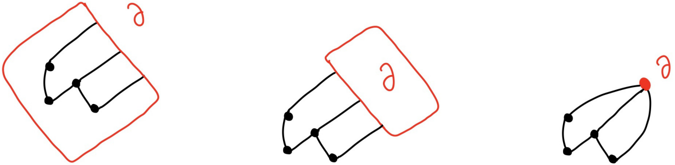

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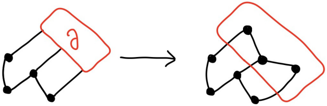
This provides:

- total graphs
- strategy to deal with the outside, and any holes in a graph

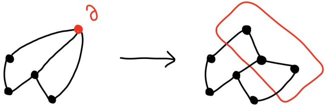
Requirements for Graph Morphisms(1)



What are embeddings?

Requirements for Graph Morphisms(1)

- vertex map needs to be partial



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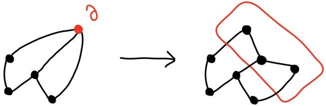
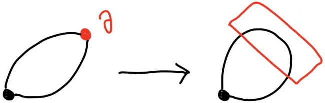


- cannot be injective on edges



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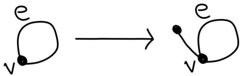
Requirements for Graph Morphisms(2)

- vertices must not change their arity



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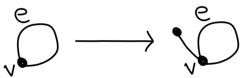
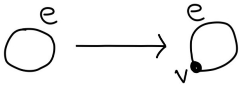
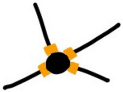


- morphisms from edges to loops are allowed

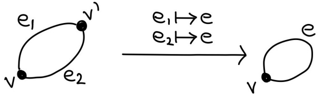
but the other way is not



Flags



- connection points between vertices and their incident edges, pairs (v, e)
- flag map (f_E, f_V) *partial* map induced by graph map



- characterize morphisms/embeddings by properties of the flag map

Graphs with Circles

Objects are total graphs, as defined above
Morphisms are (f_E, f_V) where

- the flag map is surjective
  (no increase in flags at a vertex)
- $+$ other conditions

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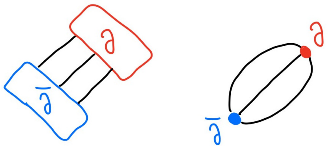
It's a category!

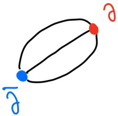
Rewriting for Graphs with Circles

specify the cases for applying a rewrite rule
Boundary graph:



Partitioning Spans

partition a graph into two (connected) parts: context and subgraph



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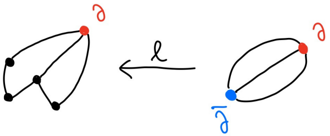

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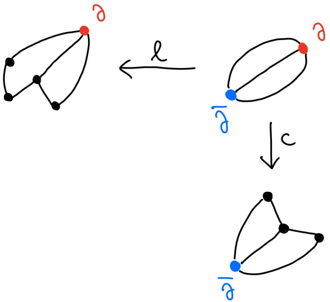

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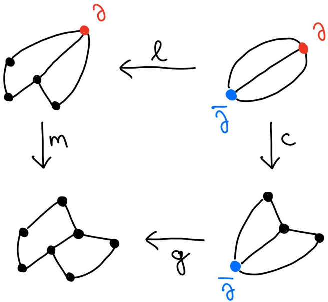
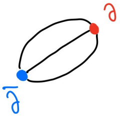


Theorem

Pushouts of partitioning spans exist, and all morphisms in the pushout square are embeddings.

Boundary Embeddings

for constructing pushout complements which give rise to partitioning spans



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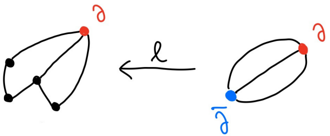

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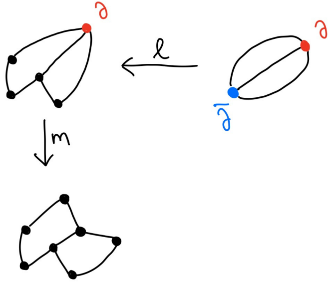

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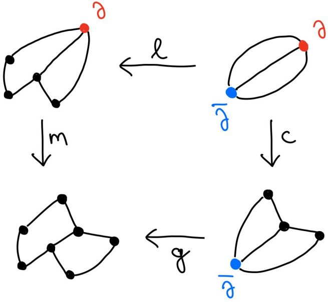
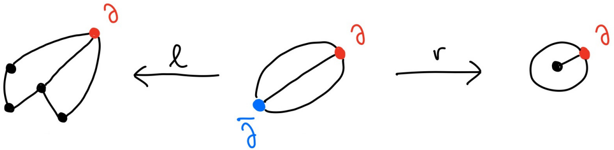


Theorem

Pushout complements of boundary embeddings exist and are unique (up to degeneracies).

An Example of Applying a Rewrite Rule



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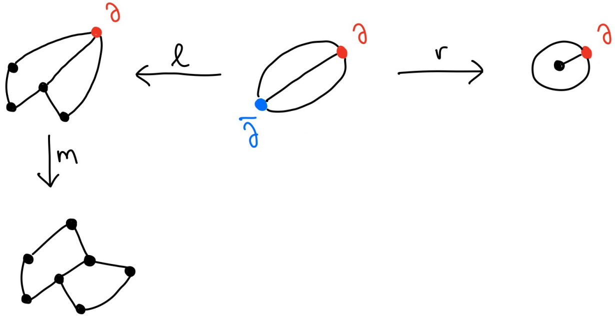

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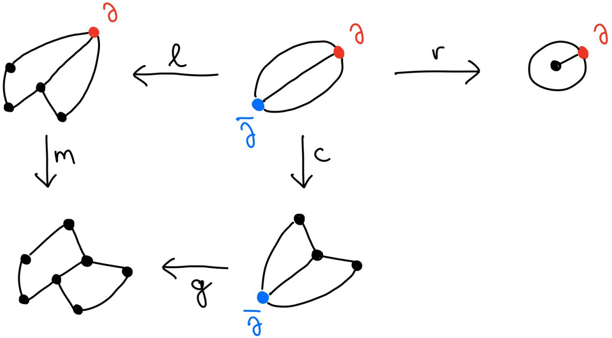

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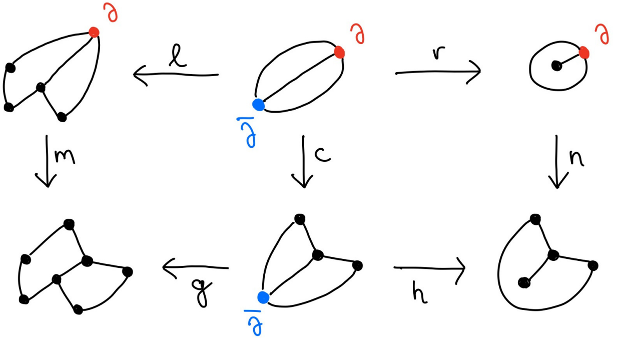

Category of Rotation Systems

so far, edges around vertices were *sets*...

- objects are rotation systems: assign to a cyclic ordering of flags to all vertices
- morphisms are morphisms in the underlying category of graphs, plus an order preservation condition

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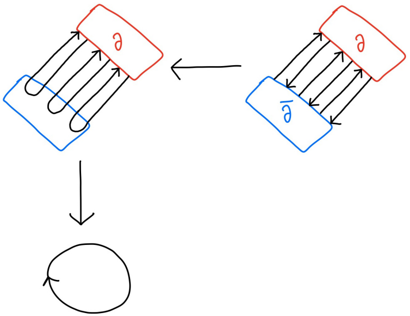
Theorem

Pushouts and pushout complements are the same as in the underlying category.

Let's talk about Loops!

problem: construct a pushout complement of a loop



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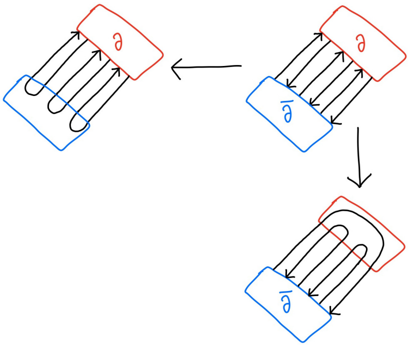

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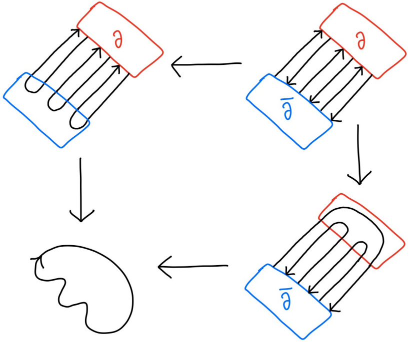

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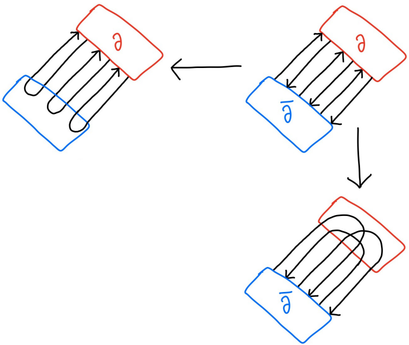

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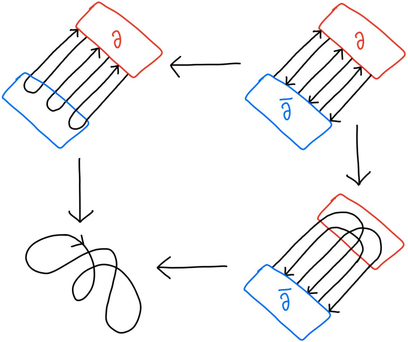

has a plane solution
and a non-plane solution

Summary

- fix inputs and outputs to control topology
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- category of graphs with circles extendable to rotation systems

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Future Thoughts

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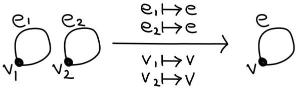
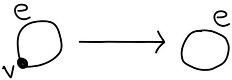
Thank You for Your Attention!

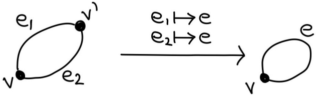
Appendix: Examples

Valid morphisms:

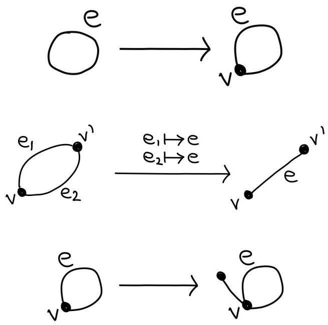


Embeddings:





Appendix: Non-Examples



Appendix: Graphs with Circles

A morphism $f : G \to G'$ between two graphs with circles consists of two (partial) functions $f_V : V \rightharpoonup V'$ as above, and $f_A : A \to A'$, satisfying the conditions listed below. Note that any such f_A factors as four maps,

$$\begin{array}{cc} f_E : E \to E' & f_{EO} : E \to O' \\ f_{OE} : O \to E' & f_O : O \to O' \end{array}$$

The following conditions must be satisfied:

- $f_A : A \to A'$ is total;
- the component $f_{OE} : O \to E'$ is the empty function;
- the pair (f_V, f_E) forms a flag surjection between the underlying graphs.

If, additionally, the following three conditions are satisfied, we call the morphism an *embedding*:

- $f_V : V \rightharpoonup V'$ is injective;
- the component f_O is injective;
- the pair (f_V, f_E) forms a flag bijection between the underlying graphs.

Appendix: Flag Surjectivity

Let $f : G \to G'$ be a morphism between two total graphs; we say that f is *flag surjective* if the two diagrams below commute laxly,

$$\begin{tikzcd} V \arrow[r, harpoon, "f_V"] \arrow[d, "s^{-1}"'] \arrow[dr, phantom, "\geq"] & V' \arrow[d, "s'^{-1}"] \\ P(E) \arrow[r, "P(f_E)"'] & P(E') \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, harpoon, "f_V"] \arrow[d, "t^{-1}"'] \arrow[dr, phantom, "\geq"] & V' \arrow[d, "t'^{-1}"] \\ P(E) \arrow[r, "P(f_E)"'] & P(E') \end{tikzcd}$$